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SCANDINAVIAN INFORMATION PROCESSING SYSTEMS

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Title: fit

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Abstract: The procedure fit computes the coefficients of a weighted least square polynomialapproximation by means of orthogonal polynomials. 8 pages.

procedure fit(i, pi, xi, yi, C, l, u)

Function.

From a table of pi, xi and yi fit computes a polynomial P(x) such that v = sum(pi×(yi-P(xi))××2) is minimum. The order, say o, of P(x) is the least number ≥ l and ≤ v for which v(o)/(n - o - 1) < v(o + 1)/(n - o).

The input table is scanned once in the order p1, x1, y1, p2, x2, y2, ...

Parameters.

integer i	supplies the number of points as input, is used as index in pi, xi and yi, and gives the order of the polynomial as output.
real pi	is an expression giving the weight of point no. i
real xi	is an expression giving the x-coordinat of point no. i.
real yi	is an expression giving the y-coordinat of point no. i.
array C(0:u)	is an array to which the coefficients of the polynomial is assigned. P(x) = C(0) + C(1)×x + C(2)×x××2 + ... C(i)×x××i.
integer l	is the lowest acceptable order of P.
integer u	is the highest acceptable order of P.

Method.

If f(0, x), f(1, x), ... f(k, x), ... is a set of nonzero polynomials of order 0, 1, ... , k, ... such that

$$\sum_{i=1}^{n} p(i) \times f(k, x(i)) \times f(l, x(i)) = 0 \text{ for } k \neq l$$

then the polynomial

$$P(k, x) = \sum_{j=0}^{k} c(j) f(j, x)$$

where

$$c(j) = \sum_{i=1}^{n} p(i) \times f(j, x(i)) \times y(i) \Big/ \sum_{i=1}^{n} p(i) \times f(j, x(i))^2$$

is a solution to the problem. A numerically better formula for c is

$$c(j) = \sum_{i=1}^{n} p(i) \times f(j, x(i)) \times (y(i) - P(j-1, x(i)) / \sum_{i=1}^{n} p(i) \times f(j,x(i))^2.$$

f(k, x) may be computed by the recurrence formula

$$f(k+1, x) := (x - a(k)) \times f(k, x) - b(k-1) \times f(k-1, x)$$

where

$$a(k) = \sum_{i=1}^{n} p(i)x(i)f(k, x(i))^2 / \sum_{i=1}^{n} p(i)f(k, x(i))^2$$

$$b(k) = \sum_{i=1}^{n} p(i)f(k, x(i))^2 / \sum_{i=1}^{n} p(i)f(k-1, x(i))^2$$

$$f(0, x) = 1; \quad f(1, x) = x - \text{Sum } p(i)x(i) / \text{Sum } p(i)$$

In the procedure

$$F(k, i) = f(k, x(i)) \times \text{sqrt}(p(i))$$

and $R(k, i) = (y(i) - P(k-1, x(i)) \text{ sqrt}(p(i))$

is used with the following start values:

$$b(-1) = 0, F(-1, i) = 0, F(0, i) = \text{sqrt}(p(i))$$

and $R(0, i) = y(i) \times \text{sqrt}(p(i))$

recurrence for k = 0, 1, ...

$$a(k) = \text{Sum } x(i)F(k, i)^2 / \text{Sum } F(k, i)^2$$

$$c(k) = \text{Sum } R(k, i)F(k, i) / \text{Sum } F(k, i)^2$$

$$R(k+1, i) = R(k, i) - F(k, i) \times c(k)$$

$$F(k+1, i) = (x(i) - a(k))F(k, i) - b(k-1)F(k-1, i)$$

$$b(k) = \text{Sum } F(k+1, i)^2 / \text{Sum } F(k, i)^2$$

From the a, b and c's the final coefficients of P(x) are computed. The recurrence formula

$$c(j, i) = c(j-1, i) - a(i-j)\times c(j, j+1) - b(i-j)\times c(j, i+2)$$

starting from

$$c(0, i) = \text{if } 0 \leq i \leq k + 1 \text{ then } c(k) \text{ else } 0$$

gives

$$P(x) = \sum_{i=0}^{k+1} c(i+1, i) \times x\times\times i.$$

we have namely

$$P(x) = \sum_{i=0}^{b+1} c(0, i) f(i, x)$$

$$= c(1, 0) + \sum_{i=1}^{b+1} c(1, i) f(i-1, x) x$$

$$= c(1, 0) + c(2, 1) x + \sum_{i=1}^{k+1} c(2, i) f(i-2, x) x^2$$

$$= \sum_{i=0}^{j-1} c(i+1, i) x^i + \sum_{i=j}^{k+1} c(j, i) f(i-j, x) x^j$$

$$= \sum_{i=0}^{b+1} c(i+1, i) x^i$$

Example.

A table of y = x××9-x××5 with weights p = x××2+1. The variance is sum(p×(y-P(x))××2))/(n-o-1)

10 points, l=6, u=8, order = 8   103.000 ms

coefficients:

-2.61591⏨-3
1.63066⏨-1
2.88315⏨-1
-1.94733⏨ 0
-6.81127⏨-1
5.13975⏨ 0
-2.37534⏨ 0
-4.90519⏨ 0
4.36197⏨ 0

p	x	v	P(x)	v-P(x)	
1.00490⏨ 0	-7.00000⏨-2	1.68066⏨-6	-1.19751⏨-2	1.2⏨-2	
1.73960⏨ 0	8.60000⏨-1	-2.13100⏨-1	-2.04917⏨-1	-8.2⏨-3	
4.20410⏨ 0	1.79000⏨ 0	1.70282⏨ 2	1.70282⏨ 2	6.8⏨-5	
1.09610⏨ 0	-3.10000⏨-1	2.83648⏨-3	1.11621⏨-2	-8.3⏨-3	
1.38440⏨ 0	6.20000⏨-1	-7.80762⏨-2	-9.69892⏨-2	1.9⏨ -2	
3.40250⏨ 0	1.55000⏨ 0	4.26933⏨ 1	4.26938⏨ 1	-5.5⏨-4	
1.30250⏨ 0	-5.50000⏨-1	4.57231⏨-2	4.33494⏨-2	2.4⏨-3	
1.14440⏨ 0	3.80000⏨-1	-7.75830⏨-3	9.78251⏨-3	-1.8⏨-2	
2.71610⏨ 0	1.31000⏨ 0	7.50371⏨ 0	7.50219⏨ 0	1.5⏨-3	
1.62410⏨ 0	-7.90000⏨-1	1.87854⏨-1	1.88112⏨-1	-2.6⏨-4	variance = 1.20⏨-3

10 points, l = 7, u = 9, order = 8   114.000 ms variance = 1.20⏨-3

10 points, l = 8, u = 9, order = 8   112.000 ms variance = 1.20⏨-3

10 points, l = 8, u = 9, order = 9   121.000 ms

coefficients:

2.79397⏨-9
2.79397⏨-9
-6.51926⏨-9
-5.82077⏨-9
1.30385⏨-8
-1.00000⏨ 0
-1.73750⏨-8
1.68802⏨-9
-0.29104⏨-9
1.00000⏨ 0

p	x	v	P(x)	v-P(x)	
1.00490⏨ 0	-7.00000⏨-2	1.68066⏨-6	1.68323⏨-6	-2.6⏨-9	
1.73960⏨ 0	8.60000⏨-1	-2.13100⏨-1	-2.13100⏨-1	-2.8⏨-9	
4.20410⏨ 0	1.79000⏨ 0	1.70282⏨ 2	1.70282⏨ 2	-7.5⏨-9	
1.09610⏨ 0	-3.10000⏨-1	2.83648⏨-3	2.83648⏨-3	-1.5⏨-9	
1.38440⏨ 0	6.20000⏨-1	-7.80762⏨-2	-7.80642⏨-2	-2.7⏨-9	
3.40250⏨ 0	1.55000⏨ 0	4.26933⏨ 1	4.26933⏨ 1	1.9⏨-9	
1.30250⏨ 0	-5.50000⏨-1	4.57231⏨-2	4.57231⏨-2	-0.4⏨-9	
1.14440⏨ 0	3.80000⏨-1	-7.75830⏨-3	-7.75830⏨-3	-2.9⏨-9	
2.71610⏨ 0	1.31000⏨ 0	7.50371⏨ 0	7.50371⏨ 0	2.8⏨-9	
1.62410⏨ 0	-7.90000⏨-1	1.87854⏨-1	1.87584⏨-1	3.7⏨-9	variance = 9.00⏨ 9

15 points, l = 6, u = 8, order = 8 145.000 ms
coefficients:
-5.63362$_{10}$-2
2.88497$_{10}$-2
1.92269$_{10}$ 0
-1.79384$_{10}$ 0
-7.03926$_{10}$ 0
8.64038$_{10}$ 0
2.95869$_{10}$ 0
-1.04081$_{10}$ 1
5.72527$_{10}$ 0

p	x	v	P(x)	v-P(x)
1.00490$_{10}$ 0	-7.00000$_{10}$-2	1.68066$_{10}$-6	-4.85023$_{10}$-2	4.9$_{10}$-2
1.73960$_{10}$ 0	8.60000$_{10}$-1	-2.13100$_{10}$-1	-2.47513$_{10}$-1	3.4$_{10}$-2
4.20410$_{10}$ 0	1.79000$_{10}$ 0	1.79282$_{10}$ 2	1.70294$_{10}$ 2	-1.2$_{10}$-2
1.09610$_{10}$ 0	-3.10000$_{10}$-1	2.83648$_{10}$-3	8.91634$_{10}$-2	-8.6$_{10}$-2
1.38440$_{10}$ 0	6.20000$_{10}$-1	-7.80762$_{10}$-2	-4.89320$_{10}$-2	-2.9$_{10}$-2
3.40250$_{10}$ 0	1.55000$_{10}$ 0	4.26933$_{10}$ 1	4.26580$_{10}$ 1	3.5$_{10}$-2
1.30250$_{10}$ 0	-5.50000$_{10}$-1	4.57231$_{10}$-2	1.71621$_{10}$-2	2.9$_{10}$-2
1.14440$_{10}$ 0	3.80000$_{10}$-1	-7.75830$_{10}$-3	5.50052$_{10}$-2	-6.3$_{10}$-2
2.71610$_{10}$ 0	1.31000$_{10}$ 0	7.50371$_{10}$ 0	7.55223$_{10}$ 0	-4.9$_{10}$-2
1.62410$_{10}$ 0	-7.90000$_{10}$-1	1.87854$_{10}$-1	1.91326$_{10}$-1	-3.5$_{10}$-3
1.01960$_{10}$ 0	1.40000$_{10}$-1	-5.37617$_{10}$-5	-2.17621$_{10}$-2	2.2$_{10}$-2
2.14490$_{10}$ 0	1.07000$_{10}$ 0	4.35907$_{10}$-1	4.34033$_{10}$-1	1.9$_{10}$-3
5.00000$_{10}$ 0	2.00000$_{10}$ 0	4.80000$_{10}$ 2	4.79998$_{10}$ 2	1.8$_{10}$-3
1.01000$_{10}$ 0	-1.00000$_{10}$-1	9.99900$_{10}$-6	-3.89867$_{10}$-2	3.9$_{10}$-2
1.68890$_{10}$ 0	8.30000$_{10}$-1	-2.06964$_{10}$-1	-2.38302$_{10}$-1	3.1$_{10}$-2

variance = 5.71$_{10}$-3

15 points, l = 7, u = 9, order = 9 163.000 ms
coefficients:
3.72529$_{10}$-9
-1.95578$_{10}$-8
3.02680$_{10}$-8
-7.89296$_{10}$-8
-6.37956$_{10}$-8
-1.00000$_{10}$ 0
-1.64146$_{10}$-7
-2.26632$_{10}$-7
2.11265$_{10}$-7
1.00000$_{10}$ 0

p	x	v	P(x)	v-P(x)
1.00490$_{10}$ 0	-7.00000$_{10}$-2	1.68066$_{10}$-6	1.68593$_{10}$-6	-5.3$_{10}$-9
1.73960$_{10}$ 0	8.60000$_{10}$-1	-2.13100$_{10}$-1	-2.13100$_{10}$-1	4.9$_{10}$-9
4.20410$_{10}$ 0	1.79000$_{10}$ 0	1.70282$_{10}$ 2	1.70282$_{10}$ 2	3.0$_{10}$-8
1.09610$_{10}$ 0	-3.10000$_{10}$-1	2.83648$_{10}$-3	2.83649$_{10}$-3	-1.3$_{10}$-8
1.38440$_{10}$ 0	6.20000$_{10}$-1	-7.80762$_{10}$-2	-7.80762$_{10}$-2	6.2$_{10}$-9
3.40250$_{10}$ 0	1.55000$_{10}$ 0	4.26933$_{10}$ 1	4.26933$_{10}$ 1	3.0$_{10}$-8
1.30250$_{10}$ 0	-5.50000$_{10}$-1	4.57231$_{10}$-2	4.57231$_{10}$-2	-1.4$_{10}$-8
1.14440$_{10}$ 0	3.80000$_{10}$-1	-7.75830$_{10}$-3	-7.75830$_{10}$-3	2.9$_{10}$-9
2.71610$_{10}$ 0	1.31000$_{10}$ 0	7.50371$_{10}$ 0	7.50371$_{10}$ 0	1.8$_{10}$-8
1.62410$_{10}$ 0	-7.90000$_{10}$-1	1.87854$_{10}$-1	1.87854$_{10}$-1	1.5$_{10}$-8
1.01960$_{10}$ 0	1.40000$_{10}$-1	-5.37617$_{10}$-5	-5.37604$_{10}$-5	-1.4$_{10}$-9
2.14490$_{10}$ 0	1.07000$_{10}$ 0	4.35907$_{10}$-1	4.35907$_{10}$-1	4.9$_{10}$-9
5.00000$_{10}$ 0	2.00000$_{10}$ 0	4.80000$_{10}$ 2	4.80000$_{10}$ 2	6.0$_{10}$-8
1.01000$_{10}$ 0	-1.00000$_{10}$-1	9.99900$_{10}$-6	1.00051$_{10}$-5	-6.1$_{10}$-9
1.68890$_{10}$ 0	8.30000$_{10}$-1	-2.06964$_{10}$-1	-2.06964$_{10}$-1	5.2$_{10}$-9

variance 0.00$_{10}$ 0

15 points, l = 8, u = 10, order = 9 176.000 ms variance = 0.00$_{10}$ 0

15 points, l = 4, u = 11, order = 9 171.000 ms variance = 0.00$_{10}$ 0
end

The employed program:

```
begin integer i, i1,n,k,j,h; real x,v,t,t0,q,s,a; array C(0:12);
  real procedure p(dum); integer dum;
  begin
    x:= (31×i) mod 101×(3/100)-1;
    y:= x××9-x××5;
    p:= 1+x××2
end p;

for n:= 10,15 do begin
  i1:= -1;
  for k:= 6 step 1 until 9 do begin
    t0:= time+25600;
    for j:= 0, j+10 while t < t0 do begin
      i:= n; fit(i,p(i),x,v,C,k,k+2); t:= time
    end;
    t:= (t-t0+25600)/j;
    write(out,<:<10><10>:>,n,<:  points, order:=>, i, <<   ddd.000>,
      t,<: ms:>);
    if i > i1 then begin
      write(out, <:<10>coefficients::>);
      for j:= 0 step 1 until i do
        write(out,<:<10>:>,<<-d.ddddd₁₀-d>, C(j));
      write(out,<:
     p            x            v           P(x)        v-P(x):>);
      end i > i1;
      s:= 0; j:= 1;
      for i:= 1 step 1 until n do begin
        q:= p(i) ; a:= C(i);
        for h:= j - 1 step -1 until 0 do a:= a×x+C(h);
        if j > i1 then write(out,<:<10>:>,<<-d.ddddd₁₀-d>,q,x,v,a,
          << -d.d₁₀-d>,v-a);
        s:= s + (y-a)××2×q
      end i;
      write(out,<<-d.dd₁₀-d>,<: variance:=>,
        if n =j+1 then 9₁₀9 else s/(n-j-1));
      i1:= j
    end k;
  end n
end
```

Time and storage

execution time: (2 + order) × (no of points) mS

program text: 29 lines on 2 segments

program code: 3 segments

variables: 25 + 8 × no. of points + 4 × order words

The procedure

```
pm   9.7.1969  19,07,58

external procedure fit(i,pi,xi,yi,C,l,u); value l,u;
      integer i,l,u; real pi,xi,yi; array C;
begin integer j,k,n;
      real fj,r,rf,f,fx,f1,a,b,c;
      array F,F1,X,R(1:i),A,B(0:u);
  n:=i;  r:=rf:=f:=fx:=b:=0;
  for i:=1 step 1 until n do begin
    fj:=F(i):=sqrt(pi);  F1(i):=0;
    X(i):=xi;
    R(i):=yi×fj;  r:=r+R(i)××2;  rf:=rf+R(i)×fj;
    f:=f+fj×fj;  fx:=fx+X(i)×fj×fj
  end i;
  for i:=0,k+1 while k<l | (i<u ∧ f×r<(n-i)×rf×rf) do begin
    k:=i;  a:=A(k):=fx/f;  c:=C(k):=rf/f;
    f1:=f;  r:=rf:=f:=fx:=0;
    for j:=1 step 1 until n do begin
      R(j):=R(j)-F(j)×c;  r:=r+R(j)××2;
      fj:=(X(j)-a)×F(j)-b×F1(j);  F1(j):=F(j);  F(j):=fj;
      rf:=rf+R(j)×fj;  f:=f+fj×fj;  fx:=fx+X(j)×fj×fj
    end j;
    b:=B(k):=f/f1
  end i;
  if f×r<(n-k-1)×rf×rf then C(k+1):=rf/f else k:=k-1;
  i:=k+1;
  for l:=0 step 1 until k do begin
    C(k):=C(k)-A(k-l)×C(k+1);
    for j:=k-1 step -1 until l do C(j):=C(j)-A(j-l)×C(j+1)-B(j-l)×C(j+2)
  end l
end fit;  end
```