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MATHEMATICAL PROCEDURE LIBRARY

Bessel Functions - Jn(x) and Yn(x)

KEYWORDS
Mathematical, complete description, algol procedure, bessel.

ABSTRACT
The procedure besseljy calculates J0(x),J1(x),...,Jn(x) and Y0(x),Y1(x),...,Yn(x).

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1. Function and Parameters.

besseljy calculates the Bessel functions of first kind J0(x),J1(x),...,Jn(x) and the Bessel functions of second kind Y0(x),Y1(x),...,Yn(x) of integer order and with real argument.

Procedure heading:

```
procedure besseljy (n,x,J,Y);
value n,x; integer n; real x; array J,Y;
```

Call Parameters:

n	(integer or real) (real is rounded to nearest integer) maximum order of the Bessel functions. n must be >=0.
x	(integer or real) the argument, must be <>0.

Return Parameters:

J	(real array J(0:n)) the calculated values of the functions J(0)=J0(x),...,J(n)=Jn(x).
Y	(real array Y(0:n)) the calculated values of the functions Y(0)=Y0(x),...,Y(n)=Yn(x).

2. Method.

First the functions Ji(x) are calculated.

For abs x <= $_{10}$-5 the procedure uses a truncated power expansion

$$J(i)(x) = (x/2)\times\times i/i$$

for i = 0,1,...,n.

For abs x > $_{10}$-5 the values are found by recurrence, see [1]

$$j(i-1)(x) = 2\times i/x\times j(i)(x)-j(i+1)(x)$$

for i = nb-2,nb-3,...,1, where j(nb)(x)=0 and j(nb-1)(x)=$_{10}$-150. nb is an even integer found as a function of x and n, within the limits given in [2].

After recurrence J(i)(x) is found as

$$J(i) = j(i)(x)/(j(0)(x)+2\times\sum_{m=1}^{nb/2} j(2\times m)(x)$$

for i = 0,1,...,n.

Then Y0(x) and Y1(x) are found from the summation theorems, see [1]:

$$Y0(x) = 2/pi \times ((gamma+ln(abs(x/2)))\times J0(x) - 4\times \sum_{p=1}^{nb/2} ((-1)\times\times p/(2\times p)\times J(2\times p)(x)))$$

$$Y1(x) = 2/pi \times (-1/x\times J0(x)+(ln(abs(x/2))+gamma-1)\times J1(x) + \sum_{o=3}^{nb/2} ((-1)\times\times((a+1)/2)\times 4\times o/(o\times\times 2-1)\times J0(x)))$$

for odd values of o. The upper bound nb/2 is a substitute for infinity, see [2].

3. Accuracy, Time and Storage Requirements.

Accuracy: relative error $<_{10}-7$

Time: if abs x $<=_{10}-5$: approx. 5 ms
else approx. 10 + 0.7×n ms.
e.g. x=2 , n=20: 17 ms
x=10, n=40: 28 ms

Storage requirements: 2 segments of program and 8 local real variables. (during translation 3 segments).

Typographical length: 75 lines incl. last comment.

4. Test and discussion.

The algorithm is similar to the GIER procedure O.No. 208, [3].

Below program gives the following output:

```
begin

comment here the procedure is copied unless it is already
translated as an external;

integer i,n; real x;
  write(out,<:<12><10> n     x :>,false add 32,13,<:J(n):>,
            false add 32,18,<:Y(n)<10>:>);
AGAIN:
  read(in,n); if n=-1 then goto END;
  begin array J,Y(0:n);
  read(in,x);
  besseljy(n,x,J,Y);
  write(out,<:<10>:>,<<dd>,n,<<dd.ddd>,x,
            <<   -d.ddddd ddddd₁₀-dd>,J(n),Y(n) );
  goto AGAIN;
  end inner block;
END:
end

data:
0, 0.001
0, 0.5
0, 5
1, 5
10, 5
20, 5
-1,
```

n	x	J(n)	Y(n)
0	0.001	9.99999 75004₁₀ -1	-4.47141 66116₁₀ 0
0	0.500	9.38469 80724₁₀ -1	-4.44518 73352₁₀ -1
0	5.000	-1.77596 77133₁₀ -1	-3.08517 62526₁₀ -1
1	5.000	-3.27579 13760₁₀ -1	1.47863 14342₁₀ -1
10	5.000	1.46780 26472₁₀ -3	-2.51291 10098₁₀ 1
20	5.000	2.77033 00515₁₀-11	-5.93396 52968₁₀ 8

5. References.

[1] Goldstein and Thaler: Recurrence Techniques for the Calculation off Bessel Functions, MTAC 13 (1959), p.102.

[2] Randels and Reeves: Notes on Emperial bounds for Generating Bessel Functions. Comm. ACM, v.1, No. 5, 3.

[3] Tove Asmussen: Bessel J and Y, Gier System Library, O.No. 208, Regnecentralen, April 1964.

[4] British Association Mathematical Tables, Vol. VI, Bessel Functions, zero and unity, Cambridge University Press, 1958.

[5] British Association Mathematical Tables, Vol. X Bessel Functions, order 2 to 20, Cambridge University Press, 1952.

6. Algorithm.

```
besseljy=set 3
besseljy=algol
external

  procedure besseljy (n,x,J,Y);
  value n,x; integer n; real x; array J,Y;
  begin integer a,nb,N;
  real j0,j1,sum,y0,y1,y2;
  boolean even;
    sum:=abs x;
    x:=j1:=x/2;
    y0:=y1:=0;
    y2:=ln(sum)-0.1159315156584;
    if sum<=₁₀-5 then
    begin
      J(0):=sum:=j0:=1;
      for nb:=1 step 1 until n do J(nb):=J(nb-1)×x/nb
    end
    else
```

```
begin
  N:=n+1;
  if n>10 then
  begin
    for N:=N-1 while (sum/N)××N<⏨-100 do J(N):=0;
    N:=N+1
  end;
  nb:=0.525×sum+13;
  nb:=2×(if nb <=N//2 then N//2+1 else nb);
  j1:=⏨-150;
  j0:=sum:=0;
  even:=false;
  a:=(-1)××((nb-2)//2);
  for nb:=nb-1 step -1 until 2 do
  begin
    if nb<N then J(nb):=if even then j0 else j1;
    if even then
    begin j1:=nb/x×j0-j1; y0:=a/nb×j0+y0 end
    else
    begin
      j0:=nb/x×j1-j0;
      y1:=a×nb/(nb××2-1)×j1+y1;
      a:=-a;
      sum:=sum+j0
    end;
    even:=¬even
  end;
  j0:=j1/x-j0;
  sum:=2×sum+j0;
  J(0):=j0/sum;
  if n>0 then J(1):=j1/sum;
  for nb:=N-1 step -1 until 2 do J(nb):=J(nb)/sum
end;
Y(0):=y0:=0.636619772367581×(4×y0+y2×j0)/sum;
if n>0 then Y(1):=y1:=0.636619772367581×
                    (-j0/x/2+(y2-1)×j1-y1×4)/sum;
for nb:=2 step 1 until n do
begin
  Y(nb):=y2:=(nb-1)/x×y1-y0;
  y0:=y1;  y1:=y2
end
end besseljy;
```

comment

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